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ORIGINAL PAPER

Hybrid Approach for the Study of Concentration of the Longitudinal Dispersion Phenomenon

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Abstract
This article deals with the study of concentration of longitudinal dispersion phenomenon, which occurs in a porous medium. The Elzaki Adomian decomposition method (EADM), which combines the Elzaki transform and Adomian decomposition method, analyzes an approximate analytical solution of the governing equation with its convergence analysis. A comparison of EADM with the variational homotopy perturbation method and new integral transform homotopy perturbation method is included for the accuracy of the method.

Keywords Elzaki Adomian decomposition method · Burger's equation · Porous medium

Mathematics Subject Classification 35G20 · 35A22 · 34K28

Introduction

Diffusion and dispersion are processes found in two separate domains—the microscopic scale diffusion domain and the macroscopic scale dispersion domain [9]. The mixing in a porous medium of 2 miscible fluids replacing each other is said to be dispersion process. It is simple to understand if a single fluid is accessible in a porous medium; otherwise, it isn't straightforward. Dispersion in the simulated cases is considered to be diffusive. That is, it may be identified with the equation of convective-diffusion. Longitudinal dispersivity is an order of magnitude higher than the dispersivity in transverse directions [23]. Dispersion issues arise in groundwater flow, chemical, and petroleum engineering, oil reservoir, the study of hydrology, etc. Dispersion is a mixed diffusion process and mechanical dispersion process. This paper analyzes the phenomenon of longitudinal dispersion in a porous medium. This phenomenon is observed in coastal regions, in which seawater slowly displaces the fresh

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water-beds. The study consider some assumptions as follows—the medium is homogeneous, the coefficient of porosity and dispersion are constant, and the velocity of inlet flow is steady.

Many scholars have already examined the physical phenomena from various angles and perspectives. Recently many researchers have also studied the solution of fractional differential equations [10,11,17,18,26]. Ebach and White [7] have considered the longitudinal dispersion phenomenon for an input concentration that changes at regular interval of time. The dispersion problem in radial flow from totally penetrating, homogenous, isotropic non-absorbing confined aquifers was presented by Herteman [14]. Bruce and street [5] found both lateral and longitudinal dispersion within semi-finite non-absorbing porous medium for a constant input concentration during a steady unidirectional flow fluid. Hunt [15] has applied the method of disturbance of lateral and longitudinal dispersion in the non-uniform flow of seepage via heterogeneous aquifers. Patel and Mehta [22] have employed a Hope-Cole transformation. Al-Niami and Rushton [3] were researching the measurement of flow in porous media against dispersion. Marino [19] has studied the analysis of flow in non-absorbing porous medium toward dispersion. Basha and Habel [4] have determined the analytical solution of the 1D transportation equation based on time. Sander and Braddock [24] have studied analytical solutions for transient, unsaturated water and pollutant transport via horizontal porous media.

This paper's primary aim is to use the Elzaki Adomian Decomposition Method [29] to achieve an approximate analytical solution to the problem. The combination of Elzaki transform [8,12,13] and Adomian decomposition method [2] is given by the EADM. Using EADM, we get an infinite series solution. Other authors [16,21,29] have studied EADM to solve different kinds of equations.

Problem Statement

The problem is to get the concentration as a function of time T, and location X as 2 miscible fluids pass through porous medium on either side of the mixed zone. At $T = 0$, mixing occurs longitudinally and transversally, and a spot of fluid owing a concentration of C_0 is administered over the process, as represented in Fig. 1. The spot also travels perpendicular to the flow within the flow direction so that it forms an ellipse for a specific concentration of C_n.

Problem Formulation

Continuity equation by Darcy's law for incompressible fluids is as follows:

$$\frac{\partial \rho}{\partial T} + \nabla.(\rho \bar{v}) = 0, \tag{1}$$

Here pore seepage velocity is given by $\bar{v}$ and density is given by ρ.

When there is no growing or reducing of the dispersing material, the diffusion equation for homogeneous medium is given as,

$$\frac{\partial C}{\partial T} + \nabla.(C\bar{v}) = \nabla.\left[\rho \bar{D} \nabla \left(\frac{C}{\rho}\right)\right], \tag{2}$$

Concentration of fluid is specified by C and coefficient of dispersion with 9 components D_{ij} is given by D. Here density is constant as we consider laminar flow in homogeneous medium. That implies

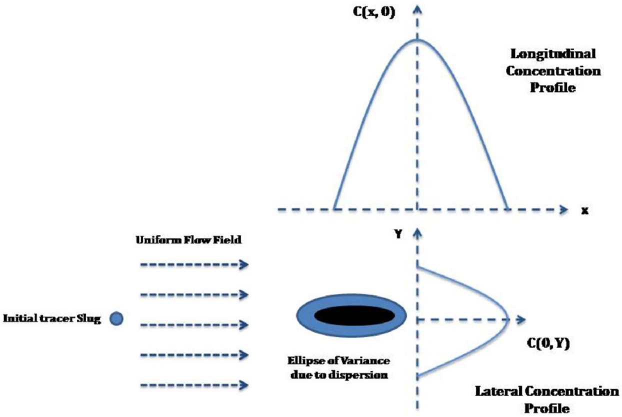

Fig. 1 Longitudinal dispersion phenomenon

$$\nabla.\bar{v} = 0, \tag{3}$$

Hence Eq. (2) is

$$\frac{\partial C}{\partial T} + \bar{v}.\nabla C = \nabla.\left(\bar{D}\nabla C\right). \tag{4}$$

It is our assumption that seepage velocity $\bar{v}$ is along the X-axis so that $\bar{v} = U(X, T)$. The coefficients of longitudinal dispersion $D_{11} \approx D_L \cong K$ are non zero and remaining are zero.

So we rewrite Eq. (4) as

$$\frac{\partial C}{\partial T} + U\frac{\partial C}{\partial X} = K\frac{\partial^2 C}{\partial X^2}, \tag{5}$$

For $X > 0$, $D_L > 0$, U can be written as $U = \frac{C(X,T)}{C_0}$ [20].

The average cross-sectional concentration is given by C. At $X = 0$, the concentration is consistent and significantly high, assuming it is $C_0 \cong 1$.

Therefore Eq. (5) is

$$\frac{\partial C}{\partial T} + C\frac{\partial C}{\partial X} = K\frac{\partial^2 C}{\partial X^2}, \tag{6}$$

Equation (6) is known as Burger's equation which expresses the longitudinal dispersion.

Here initial and boundary conditions are

$$\begin{aligned} C(X, 0) &= e^{-X}, \ X \geq 0, \\ C(0, T) &= 1, \ \ 0.001 \leq T \leq 0.01. \end{aligned} \tag{7}$$

Here concentration decreases with distance X, so that the function is considered as negative exponential for simplicity [20].

Implementation of Elzaki Adomian Decomposition Method

Applying Elzaki transform to equation (6) and using differential properties of transform, we get

$$\frac{E\left[C(X,T)\right]}{v} - vC(X,0) + E\left[C.C_X\right] = E\left[K\,C_{XX}\right],$$

$$E\left[C(X,T)\right] - v^2 e^{-X} + vE\left[C.C_X\right] = vE\left[K\,C_{XX}\right], \tag{8}$$

Using inverse of Elzaki transform on equation (8),

$$C(X,T) = e^{-X} - E^{-1}\left[vE\left\{C.C_X\right\}\right] + E^{-1}\left[vE\left\{K\,C_{XX}\right\}\right], \tag{9}$$

By applying the decomposition approach,

$$\sum_{n=0}^{\infty} C_n(X,T) = e^{-X} - E^{-1}\left[vE\left\{\sum_{n=0}^{\infty} A_n\right\}\right] + E^{-1}\left[vE\left\{K\sum_{n=0}^{\infty} C_{nXX}\right\}\right], \tag{10}$$

Collating results on each side of Eq. (10) and $K = 1$ [25],

$$\begin{aligned}
&C_0(X,T) = e^{-X},\\
&C_1(X,T) = -E^{-1}\left[vE\left(A_0\right)\right] + E^{-1}\left[vE\left(C_{0XX}\right)\right],\\
&C_2(X,T) = -E^{-1}\left[vE\left(A_1\right)\right] + E^{-1}\left[vE\left(C_{1XX}\right)\right],\\
&C_3(X,T) = -E^{-1}\left[vE\left(A_2\right)\right] + E^{-1}\left[vE\left(C_{2XX}\right)\right],\\
&C_4(X,T) = -E^{-1}\left[vE\left(A_3\right)\right] + E^{-1}\left[vE\left(C_{3XX}\right)\right],\\
&\ldots
\end{aligned} \tag{11}$$

where A_n is the Adomian polynomial which shows the nonlinear term $(C.C_X)$ which can be computed using the formula described in [1]. Few components of Adomian polynomials are,

$$\begin{aligned}
&A_0 = C_0 C_{0X},\\
&A_1 = C_0 C_{1X} + C_1 C_{0X},\\
&A_2 = C_0 C_{2X} + C_1 C_{1X} + C_2 C_{0X},\\
&A_3 = C_{1X} C_2 + C_1 C_{2X} + C_{0X} C_3 + C_0 C_{3X},\\
&\ldots
\end{aligned} \tag{12}$$

Employing Adomian polynomials (12) and the iteration formulas (11),

$$\begin{aligned}
&C_0(X,T) = e^{-X},\\
&C_1(X,T) = \left(e^{-X} + e^{-2X}\right)T,\\
&C_2(X,T) = \left(e^{-X} + 6e^{-2X} + 3e^{-3X}\right)\frac{T^2}{2},\\
&C_3(X,T) = \left(e^{-X} + 28e^{-2X} + 51e^{-3X} + 16e^{-4X}\right)\frac{T^3}{6},\\
&C_4(X,T) = \left(e^{-X} + 120e^{-2X} + 606\,e^{-3X} + 568e^{-4X} + 125e^{-5X}\right)\frac{T^4}{24},\\
&\ldots
\end{aligned} \tag{13}$$

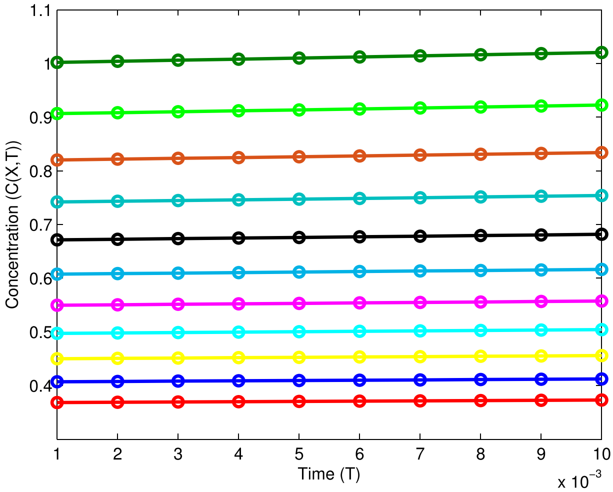

Fig. 2 Concentration versus time for various values of T

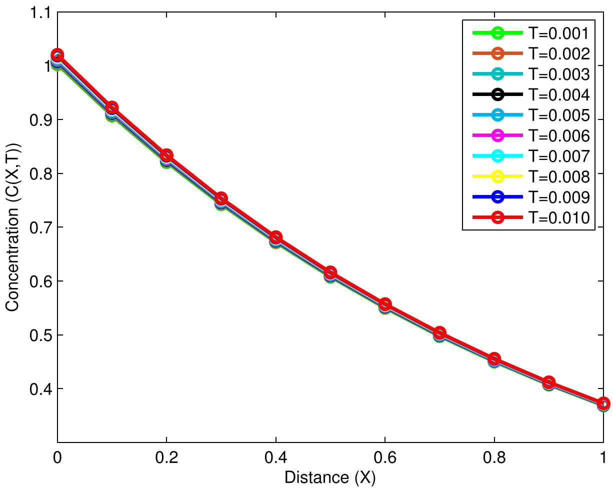

Fig. 3 Concentration versus distance for various values of X

The approximate solution is

$$\begin{aligned} C(X,T) = e^{-X} &+ \left(e^{-X} + e^{-2X}\right)T + \left(e^{-X} + 6e^{-2X} + 3e^{-3X}\right)\frac{T^2}{2} \\ &+ \left(e^{-X} + 28e^{-2X} + 51e^{-3X} + 16e^{-4X}\right)\frac{T^3}{6} \\ &+ \left(e^{-X} + 120e^{-2X} + 606\,e^{-3X} + 568e^{-4X} + 125e^{-5X}\right)\frac{T^4}{24} + \cdots \end{aligned} \tag{14}$$

which is the solution of equation (6). The numerical solution of equation (14) is shown in Table 1 and its graphical representations are shown in Figs. 2, 3, 4 and 5.

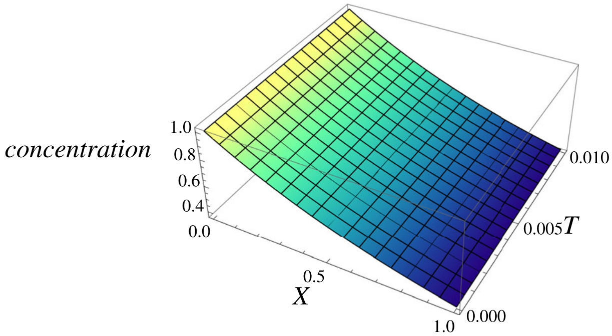

Fig. 4 3D behaviour of concentration

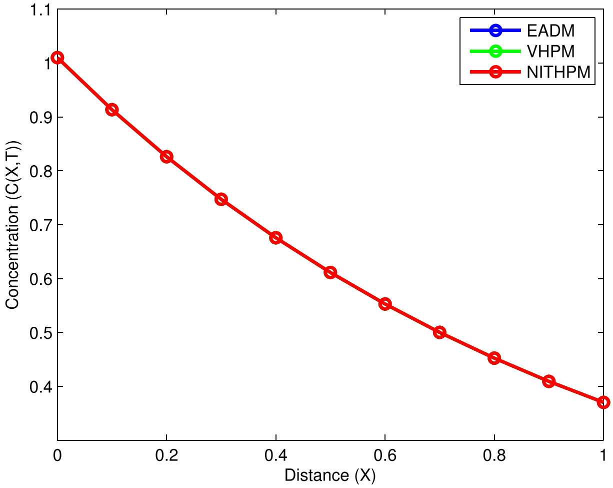

Fig. 5 Comparison of solutions by EADM, VHPM and NITHPM for fixed time $T = 0.005$

Applying Corollary [27,28] for convergence analysis, we have

$$\begin{aligned} \gamma_0 &= \frac{\|C_1\|}{\|C_0\|} = 0.0136788 < 1, \\ \gamma_1 &= \frac{\|C_2\|}{\|C_1\|} = 0.0132076 < 1, \\ \gamma_2 &= \frac{\|C_3\|}{\|C_2\|} = 0.0175273 < 1, \\ \gamma_3 &= \frac{\|C_4\|}{\|C_3\|} = 0.0207543 < 1, \\ &\dots \end{aligned} \tag{15}$$

Hence, we can say that $\sum_{i=0}^{\infty} C_i$ is convergent. Therefore the approximate solutions given by equation (15) is convergent.

Table 1 The value of concentration for various distance X and time $T = 0.001, 0.002, 0.003, \ldots 0.010$

X	$T = 0.001$	$T = 0.002$	$T = 0.003$	$T = 0.004$	$T = 0.005$
0.1	0.906565	0.908301	0.910045	0.911797	0.913557
0.2	0.820223	0.821722	0.823227	0.824739	0.826258
0.3	0.74211	0.743408	0.744711	0.746019	0.747333
0.4	0.671442	0.672568	0.673698	0.674833	0.675972
0.5	0.607507	0.608486	0.60947	0.610456	0.611447
0.6	0.549663	0.550517	0.551375	0.552235	0.553098
0.7	0.49733	0.498076	0.498825	0.499577	0.500331
0.8	0.449981	0.450635	0.451291	0.451949	0.452609
0.9	0.407142	0.407717	0.408293	0.40887	0.409449
1	0.368383	0.368889	0.369395	0.369903	0.370412
X	$T = 0.006$	$T = 0.007$	$T = 0.008$	$T = 0.009$	$T = 0.010$
0.1	0.915326	0.917104	0.91889	0.920684	0.922488
0.2	0.827784	0.829316	0.830856	0.832402	0.833955
0.3	0.748652	0.749977	0.751307	0.752643	0.753984
0.4	0.677116	0.678264	0.679417	0.680574	0.681735
0.5	0.612441	0.613438	0.61444	0.615444	0.616453
0.6	0.553964	0.554833	0.555705	0.55658	0.557458
0.7	0.501087	0.501846	0.502607	0.503371	0.504137
0.8	0.453272	0.453936	0.454602	0.45527	0.45594
0.9	0.41003	0.410612	0.411197	0.411782	0.41237
1	0.370923	0.371435	0.371948	0.372463	0.372979

Numerical Results and Discussion

Numerical and graphical representation of equation (14) is successfully obtained by using EADM. Table 1 represents the numerical results of $C_n(X, T)$ for various values of X & $T = 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.010$. Figures 2 and 3 indicate the graph of concentration versus time and concentration versus distance respectively. Figure 4 shows 3D behavior of concentration. Table 2 indicates the comparison of Elzaki Adomian Decomposition Method (EADM), Variational Homotopy Perturbation Method (VHPM) and New integral transform homotopy perturbation method (NITHPM). Figure 5 shows the comparison between the results acquired by various standard methods.

Conclusion

Here we have successfully applied EADM method to find an approximate solution of longitudinal dispersion phenomenon along with its convergence analysis. Also, from Table 1 it can be concluded that concentration decreases with distance of X and increases slightly with time T. We also presume from Table 2 that there is an excellent agreement between EADM, VHPM, and NITHPM. Analytical expressions are useful to study the accumulation of salinity in groundwater and useful for predicting the possible pollution of groundwater.

Table 2 Comparison of numerical values of concentration at different time T by EADM, VHPM and NITHPM

X	$T=0.001$			$T=0.002$			$T=0.003$		
	EADM	VHPM [6]	NITHPM [25]	EADM	VHPM[6]	NITHPM [25]	EADM	VHPM[6]	NITHPM [25]
0	1.00201	1.00201	1.00201	1.00402	1.00402	1.00402	1.00605	1.00605	1.00605
0.1	0.906565	0.90656	0.906565	0.908301	0.9083	0.908301	0.910045	0.910045	0.910045
0.2	0.820223	0.820223	0.820873	0.821722	0.82172	0.821722	0.823227	0.82322	0.823227
0.3	0.74211	0.74211	0.74211	0.743408	0.7434	0.743408	0.744711	0.74471	0.744711
0.4	0.671442	0.64144	0.671442	0.672568	0.67256	0.672568	0.673698	0.67369	0.673698
0.5	0.607507	0.6075	0.607507	0.608486	0.60848	0.608486	0.60947	0.60947	0.60947
0.6	0.549663	0.54966	0.549663	0.550517	0.55051	0.550517	0.551375	0.55137	0.551375
0.7	0.49733	0.49733	0.49733	0.498076	0.49807	0.498076	0.498825	0.49882	0.498825
0.8	0.449981	0.44998	0.443981	0.450635	0.45063	0.450635	0.451291	0.45129	0.451291
0.9	0.407142	0.40714	0.407142	0.407717	0.40771	0.407717	0.408293	0.40829	0.408293
1	0.368383	0.36838	0.368303	0.368889	0.36888	0.368889	0.369395	0.36939	0.369395

X	$T=0.004$			$T=0.005$		
	EADM	VHPM[6]	NITHPM [25]	EADM	VHPM[6]	NITHPM [25]
0	1.00808	1.00808	1.00808	1.01013	1.01013	1.01013
0.1	0.911797	0.91179	0.911797	0.913557	0.91355	0.913557
0.2	0.824739	0.82473	0.824739	0.826258	0.82625	0.826258
0.3	0.746019	0.74601	0.746019	0.747333	0.74733	0.747333
0.4	0.674833	0.67483	0.674833	0.675972	0.67597	0.675972
0.5	0.610456	0.61045	0.610456	0.611447	0.61144	0.611447
0.6	0.552235	0.55223	0.552235	0.553098	0.55309	0.553098
0.7	0.499577	0.49957	0.499877	0.500331	0.50033	0.500331
0.8	0.451949	0.45194	0.451949	0.452609	0.452609	0.452609
0.9	0.40887	0.40887	0.40887	0.409449	0.40944	0.409449
1.0	0.369903	0.3699	0.369903	0.370412	0.37041	0.370412

Thus, the proposed method is extremely reliable and efficient to find the solution of physical phenomenon.

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Author Contributions All authors read and approved the final manuscript.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest regarding proposed manuscript.

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